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THERMAL INSTABILITY OF BLASIUS FLOW ALONG HORIZONTAL PLATES

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Abstract—The thermal instability of laminar forced convection flow along a horizontal semi-infinite flat plate heated isothermally from below or cooled isothermally from above is investigated for disturbances in the form of stationary longitudinal vortices which are periodic in the spanwise direction. The analysis uses non-parallel flow model considering the variation of the basic flow and temperature fields with the streamwise coordinate as well as the transverse velocity component in the disturbance equations. The critical values of the Grashof number $Gr_L^* = Gr_X^*/Re_X^{3/2}$ are obtained for Prandtl numbers ranging from 10^{-2} to 10^4. The Prandtl and Reynolds numbers effects on vortex-type instability for Blasius flow along horizontal plates are clarified.

NOMENCLATURE

- a, wave number, $2\pi/\lambda$;
- D, $d/d\eta$;
- F, $(\eta f' - f)/2$;
- f, dimensionless stream function;
- g, gravitational acceleration;
- G, eigenvalue, Gr_L/Re_L;
- Gr_L, Grashof number based on L, $g\beta(\Delta T)L^3/\nu^2$;
- Gr_X, Grashof number based on X, $g\beta(\Delta T)X^3/\nu^2$;
- L, characteristic length, $(\nu X/U_\infty)^{1/2}$;
- M, number of divisions in y direction;
- P, pressure;
- Pr, Prandtl number, ν/α;
- p, dimensionless pressure, $P'/(\rho U_\infty^2/Re_L)$;
- Re_L, Re_X, Reynolds numbers, $(U_\infty L/\nu) = Re_X^{1/2}$ and $(U_\infty X/\nu)$, respectively;
- T, temperature;
- U, V, W, velocity components in X, Y, Z directions;
- u, v, w, dimensionless perturbation velocities, $(U', V', W')/U_\infty$;
- X, Y, Z, rectangular coordinates;
- x, y, z, dimensionless coordinates, $(X, Y, Z)/L$.

Greek symbols

- α, thermal diffusivity;
- β, coefficient of thermal expansion;
- η, similarity variable, $Y/L = Y(U_\infty/\nu X)^{1/2} = y$;
- θ, dimensionless temperature disturbance, $\theta'/\Delta T$;
- λ, dimensionless wavelength of vortex rolls, $2\pi/a$;
- ν, kinematic viscosity;
- ρ, density;
- τ, dimensionless temperature, $(T_b - T_\infty)/\Delta T$;
- ΔT, temperature difference, $(T_w - T_\infty)$.

Subscripts and Superscripts

- $*$, critical value or dimensionless disturbance amplitude;
- $'$, prime, disturbance quantity or differentiation with respect to η;
- b, basic flow quantity;
- w, value at wall;
- ∞, free stream condition.

1. INTRODUCTION

BUOYANCY effects in laminar forced convective flow over a heated horizontal semi-infinite flat plate were first studied by Mori [1] and Sparrow and Minkowycz [2] independently. These early studies apparently motivated further investigations [3–6] in recent years. When a horizontal laminar boundary layer is heated from below or cooled from above, the layer is potentially unstable because of its top-heavy situation due to the density variation of fluid with temperature. The situation is somewhat analogous to the thermal instability of plane Poiseuille flow [7–10] or the well-known Görtler instability of curved boundary layers [11]. The problem of hydrodynamic stability for the laminar boundary layer involving the solution of Orr–Sommerfeld equation has been studied rather extensively in the past. In contrast, the thermal instability problem does not appear to have been reported in the literature.

The purpose of this study is to determine theoretically the conditions marking the onset of longitudinal vortex rolls in a horizontal Blasius flow where the flat plate is heated isothermally from below or cooled isothermally from above. After the onset of vortex rolls, the flow and temperature fields assume a three-dimensional character and the existing flow and heat-transfer results for laminar forced convection over a flat plate may no longer apply. It is then obvious that the present problem is of considerable practical interest.

2. THE BASIC FLOW

Consideration is given to a horizontal laminar boundary-layer flow with free stream velocity U_∞ and free stream temperature T_∞ along a flat plate where the wall temperature $T_w(> T_\infty)$ is constant. The laminar forced convection flow problem is governed by the following set of equations [12]

$$f''' + \tfrac{1}{2} f f'' = 0, \tag{1}$$

$$\tau'' + \tfrac{1}{2} Pr f \tau' = 0 \tag{2}$$

with the boundary conditions

$$f(0) = f'(0) = \tau(\infty) = 0, \quad f'(\infty) = \tau(0) = 1 \tag{3}$$

where the Blasius similarity variable is

$$\eta = Y(U_\infty/\nu X)^{1/2} \equiv Y/L(X)$$

with $L(X) = (\nu X/U_\infty)^{1/2}$, the stream function $\psi = (\nu X U_\infty)^{1/2} f(\eta)$, the normalized temperature $\tau(\eta) = (T_b - T_\infty)/(T_w - T_\infty)$ and $Pr = \nu/\alpha =$ Prandtl number. Equation (1) is solved by the fourth order Runge–Kutta method and the temperature distribution τ is

$$\tau(\eta) = 1 - \frac{\int_0^\eta \left[\exp\left(-\frac{Pr}{2}\int_0^\eta f\,d\eta\right)\right] d\eta}{\int_0^\infty \left[\exp\left(-\frac{Pr}{2}\int_0^\eta f\,d\eta\right)\right] d\eta}. \tag{4}$$

The basic flow is a two-dimensional boundary-layer flow which depends on streamwise and transverse directions.

3. THE THERMAL INSTABILITY PROBLEM

To study the vortex instability of the basic Blasius flow heated from below (or cooled from above), the perturbation quantities are superimposed on the basic quantities as

$$U = U_b(X, Y) + U'(Y, Z), \quad V = V_b(X, Y) + V'(Y, Z),$$
$$W = W'(Y, Z), \quad T = T_b(X, Y) + \theta'(Y, Z),$$
$$P = P_b - \rho_\infty g Y + P'(Y, Z).$$

As discussed in [13, 14], all the flow disturbance quantities are taken to be a function of Y and Z only for neutral stability involving Görtler vortices. Further details regarding the assumed form of disturbances and some experimental fact are explained clearly in [13]. After applying the linear stability theory and using Boussinesq approximation, the perturbation equations referring to the coordinate system shown in Fig. 1 become

$$\frac{\partial V'}{\partial Y} + \frac{\partial W'}{\partial Z} = 0, \tag{6}$$

$$U' \frac{\partial U_b}{\partial X} + V_b \frac{\partial U'}{\partial Y} + V' \frac{\partial U_b}{\partial Y} = \nu \nabla_1^2 U', \tag{7}$$

$$V_b \frac{\partial V'}{\partial Y} + V' \frac{\partial U_b}{\partial Y} = -\frac{1}{\rho}\frac{\partial P'}{\partial Y} + \nu \nabla_1^2 V' + g\beta\theta', \tag{8}$$

$$V_b \frac{\partial W'}{\partial Y} = -\frac{1}{\rho}\frac{\partial P'}{\partial Z} + \nu \nabla_1^2 W', \tag{9}$$

$$U' \frac{\partial T_b}{\partial X} + V' \frac{\partial T_b}{\partial Y} + V_b \frac{\partial \theta'}{\partial Y} = \alpha \nabla_1^2 \theta', \tag{10}$$

where $\nabla_1^2 = \partial^2/\partial Y^2 + \partial^2/\partial Z^2$ and the terms involving V_b, $\partial U_b/\partial X$ and $\partial T_b/\partial X$ are retained. The term $U'\partial V_b/\partial X$ is neglected following the boundary-layer approximation $\partial V_b/\partial X \approx 0$. The nonparallelism of the basic flow is found to be important in recent investigations [13, 14] dealing with the vortex instability of natural convection flow on inclined isothermal plates. The basic flow and temperature quantities can be written in the following form

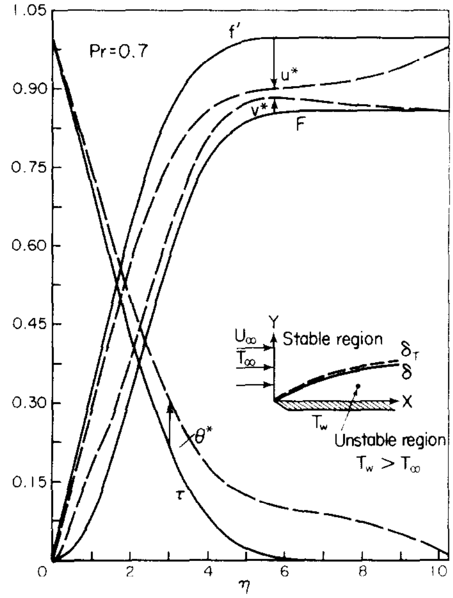

FIG. 1. Coordinate system and distributions of basic quantities f', F, τ and perturbation amplitudes u^*, v^*, θ^* for $Pr = 0.7$.

$$U_b = U_\infty f'(\eta), \; V_b = (U_\infty/2Re_L)(\eta f' - f) = (U_\infty/Re_L)F,$$
$$T_b = T_\infty + \Delta T \tau(\eta) \tag{11}$$

where $Re_L = U_\infty L/\nu = (U_\infty X/\nu)^{1/2} = Re_X^{1/2}$, $F = (\eta f' - f)/2$ and $\Delta T = T_w - T_\infty$.

After introducing the following dimensionless variables, $(x, y = \eta, z) = (X, Y, Z)/L(X)$, $(u, v, w) = (U', V'W')/U_\infty$, $p = P'/(\rho U_\infty^2/Re_L)$, $\theta = \theta'/\Delta T$ the disturbance equations can be recast into the dimensionless form as

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} = 0, \tag{12}$$

$$F\frac{\partial u}{\partial y} - \tfrac{1}{2}\eta f'' u + Re_L f'' v = \nabla^2 u, \tag{13}$$

$$F\frac{\partial v}{\partial y} + \tfrac{1}{2}\eta f'' v = \nabla^2 v - \frac{\partial p}{\partial y} + G\theta, \tag{14}$$

$$F\frac{\partial w}{\partial y} = \nabla^2 w - \frac{\partial p}{\partial z}, \tag{15}$$

$$F\frac{\partial \theta}{\partial y} - \tfrac{1}{2}\eta \tau' u + Re_L \tau' v = \frac{1}{Pr}\nabla^2 \theta, \tag{16}$$

where

$$G = g\beta(\Delta T) L Re_L/U_\infty^2 = Gr_L/Re_L,$$
$$Gr_L = g\beta(\Delta T)L^3/\nu^2 = Gr_X/Re_X^{3/2},$$
$$Gr_X = g\beta(\Delta T)X^3/\nu^2 \text{ and } \nabla^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2.$$

Upon eliminating the dependent variable w and the pressure terms from equations (14) and (15) using continuity equation (12), one obtains

$$\nabla^2 u - F\frac{\partial u}{\partial y} + \tfrac{1}{2}\eta f'' u = Re_L f'' v, \tag{17}$$

$$\nabla^2\nabla^2 v - F\frac{\partial}{\partial y}\nabla^2 v - \tfrac{1}{2}\eta f''\nabla^2 v = -G\frac{\partial^2\theta}{\partial z^2}, \tag{18}$$

$$\nabla^2\theta - PrF\frac{\partial\theta}{\partial y} = Pr\tau'(Re_L v - \tfrac{1}{2}\eta u). \tag{19}$$

The boundary conditions are $u = v = v' = \theta = 0$ at $y = 0$ and ∞. For the stationary longitudinal vortices which are periodic in the spanwise direction and neglecting the x-dependences [13, 14] at the neutral stability, the following disturbance forms are applicable.

$$u = u^+(y)\exp(iaz),\quad v = v^+(y)\exp(iaz),\quad \theta = \theta^+(y)\exp(iaz). \tag{20}$$

The quantity a is the wave number of the disturbance. Substituting equation (20) into the perturbation equations (17)–(19), the following set of equations results.

$$[(D^2-a^2) - FD + \tfrac{1}{2}\eta f'']u^+ = Re_L f'' v^+, \tag{21}$$

$$[(D^2-a^2)^2 - F(D^3 - a^2 D) - \tfrac{1}{2}\eta f''(D^2-a^2)]v^+ = a^2 G\theta^+, \tag{22}$$

$$[(D^2-a^2) - PrFD]\theta^+ = Pr\tau'(Re_L v^+ - \tfrac{1}{2}\eta u^+), \tag{23}$$

where $D = \mathrm{d}/\mathrm{d}y$. By setting $u^+ = u^*$, $Re_L v^+ = v^*$, $\theta^+ = \theta^*$ and $GRe_L = Gr_L$, the parameter Re_L does not appear explicitly and the resulting system of equations becomes

$$[(D^2-a^2) - FD + \tfrac{1}{2}\eta f'']u^* = f'' v^*, \tag{24}$$

$$[(D^2-a^2)^2 - F(D^3 - a^2 D) - \tfrac{1}{2}\eta f''(D^2-a^2)]v^* = a^2 Gr_L\theta^*, \tag{25}$$

$$[(D^2-a^2) - PrFD]\theta^* = Pr\tau'(v^* - \tfrac{1}{2}\eta u^*). \tag{26}$$

The boundary conditions are

$$u^* = v^* = Dv^* = \theta^* = 0 \text{ at } \eta = 0 \text{ and } \infty. \tag{27}$$

For the conventional parallel flow assumption for the basic flow, the terms involving F as well as the x-derivatives of the basic quantities are neglected. In the disturbance equations, the terms on the R.H.S. may be regarded as the driving terms. Equations (24)–(27) form an eigenvalue problem and the solution will be effected by a numerical method.

4. METHOD OF SOLUTION

The fourth-order finite-difference scheme used in this study is due to Thomas [15] in his study on the stability of plane Poiseuille flow and the detailed derivations are given by Chen [16] in a study on the hydrodynamic stability of developing flow in a parallel-plate channel. In the present finite-difference solution, a finite value of η must be prescribed to satisfy the boundary conditions at $\eta = \infty$ [17]. For this purpose, two cases are considered depending on the value of Prandtl number. When $Pr \geqslant 1$, the condition at infinity for θ^* is replaced by $\theta^* = 0$ at $\eta = \eta_1$ corresponding to $\tau \leqslant 10^{-8}$ since as $\tau \to 0$ one has $\tau' \to 0$. Equation (26) reveals that the flow field is stable for the region $\eta = \eta_1 \sim \infty$. On the other hand, when $Pr < 1$ the boundary condition $u^* = 0$ is set at $\eta = \eta_2$ corresponding to $(f'-1) \leqslant 10^{-8}$ and the conditions $v^* = \theta^* = 0$ are set at $\eta = \eta_1 (> \eta_2)$ corresponding to $\tau \leqslant 10^{-8}$. As $f' \to 1$, one has $f'' \to 0$ and equation (24) shows that $u^* = 0$ for the region $\eta = \eta_1 \sim \infty$. Noting that with $Pr < 1$ the thickness of the thermal boundary layer is larger than that of the hydrodynamic boundary layer, one obtains $v^* = \theta^* = 0$ for the region $\eta = \eta_1 \sim \infty$ from equations (25) and (26). The satisfactory values for the step size Δy, the number of divisions M and the end position η_1 for various Prandtl numbers are found by numerical experiments and the results are listed in Table 1 with η_2 fixed at $\eta_2 = 10.4$.

Table 1. Numerical data for Δy M and η_1

Pr	0.01	0.1	0.7	1.0	10	10^2	10^3
Δy	0.04	0.04	0.04	0.04	0.02	0.01	0.01
M	1600	650	275	260	250	200	105
η_1	64	24	11	10.4	5.0	2.0	1.05

The finite-difference technique transforms equation (25) and its boundary conditions into a quidiagonal system of matrix for a set of algebraic equations and similarly two diagonal systems result from equations (24) and (26) and their boundary conditions. The numerical solutions of the quidiagonal and tridiagonal systems are reported in [18] and [19], respectively, and will not be elaborated here.

The iterative procedure for the simultaneous solution of the three perturbation equations consists of the following main steps:

(1) With the basic velocity and temperature given, a value of the wavenumber is selected for a particular Prandtl number.

(2) The initial values for the eigenvalue Gr_L and the disturbance velocity v_k^* in the vertical direction are assigned. The selection of the initial value for v_k^* should correspond to the primary mode of disturbance. In this study, $v_k^* = 2(1-k/M)$, $k = 2, 3, \ldots, M$, is used. However, one may note that the initial disturbance in the form of $v_k^* = \sin[(k-1)\pi/M]$, $k = 1, 2, \ldots, M+1$, also leads to a satisfactory result. Any arbitrary form of the disturbance profile satisfying the boundary conditions may be used but the profiles mentioned above are found to yield a faster convergence.

(3) The finite-difference form of equation (24) is solved to obtain u_k^*.

(4) With v_k^* and u_k^* known, the finite-difference form of equation (26) is solved to obtain θ_k^*.

(5) The R.H.S. of equation (25) is now known, and new values of v_k^* are obtained by the finite-difference solution of equation (25).

(6) A new and improved eigenvalue can now be computed by the following equation [20].

$$(Gr_L)_{\text{new}} = (Gr_L)_{\text{old}}\frac{|\sum_k (v_k^*)^2_{\text{old}}|^{1/2}}{|\sum_k (v_k^*)^2_{\text{new}}|^{1/2}}. \tag{27}$$

The magnitude of the quantity v_k^* is readjusted by the following equation in order to return to the original order of magnitude.

$$v_k^* = (v_k^*)_{\text{new}}(Gr_L)_{\text{new}}/(Gr_L)_{\text{old}}. \tag{28}$$

It is well to note that the absolute value for v_k^* cannot be determined from the linearized theory and the correct profile satisfying the governing equation is sought.

(7) The steps (3)–(6) are repeated until the following convergence criteria are satisfied.

$$\varepsilon_1 = \sum_k |(v_k^*)_{\text{new}} - (v_k^*)_{\text{old}}| / \sum_k (v_k^*)_{\text{new}}| \leqslant 10^{-6}, \tag{29}$$

$$\varepsilon_2 = |(Gr_L)_{\text{new}} - (Gr_L)_{\text{old}}|/(Gr_L)_{\text{new}} \leqslant 10^{-8}. \tag{30}$$

Numerical experiments show that only a few iterations are required to satisfy the above conditions.

By varying the wavenumber a and carrying out the above iterative procedure, a minimum eigenvalue Gr_L^*, which permits a solution of the set of the disturbance equations, can be found. The minimum eigenvalue and the corresponding wavenumber are the critical values which correspond to the onset of instability.

5. THE NEUTRAL STABILITY RESULTS AND DISCUSSION

5.1. *Perturbed velocity and temperature fields*

Although the primary objective of this investigation is to obtain the critical value of the eigenvalue for the onset of stationary longitudinal rolls which are periodic in the spanwise direction, a study of the perturbed velocity and temperature fields may provide some insight into the physical mechanism of thermal instability. Figures 1 and 2 show the distributions of the basic profiles for f', F and τ with the disturbance amplitudes u^*, v^* and θ^* superimposed for the cases of $Pr = 0.7$ and 10, respectively. Since the magnitudes of the disturbance quantities cannot be determined by using the linear stability theory, the magnitude of the maximum disturbance quantity is taken to be 0.1 in the plotting. In order to study the decay of the disturbance quantity in the vertical direction, the distributions of the disturbances are also shown in Fig. 3 for $Pr = 0.7$ and 10 where the largest magnitude of the disturbances u^*, v^* and θ^* is again taken to be 0.1. It is noted that the horizontal disturbance velocity u^* is negative suggesting that the secondary flow also derives its energy from the main flow through mutual interactions as represented by the second and third terms on the L.H.S. of equation (25). The profiles for u^* and θ^* are seen to be qualitatively similar.

The secondary flow pattern at the onset of instability is of considerable interest. For this purpose one may define a stream function ψ with $v = \partial\psi/\partial z$ and $w = -\partial\psi/\partial y$ satisfying the continuity equation $\partial v/\partial y + \partial w/\partial z = 0$. From the normal modes of the disturbances, one has $v = v^+(y)e^{iaz}$ and $\psi = \psi^+(y)e^{iaz}$. Using $v = \partial\psi/\partial z$ and $v^* = Re_L v^+$, one obtains $\psi = -[iv^*(y)/a]\,e^{iaz}$. The physical meaning is attached only to the real part of the stream function and the contour lines are shown in Figs. 4 and 5 for $Pr = 0.7$ and 10, respectively. It is noted that the dimensionless wavelength is $\lambda = 2\pi/a$ and $\psi_{\max}$ is taken to be one. One immediately notices the striking resemblance between the streamline pattern of vortex disturbance for flow over concave wall [11] and the present secondary streamline pattern caused by buoyancy forces as illustrated in Figs. 4 and 5.

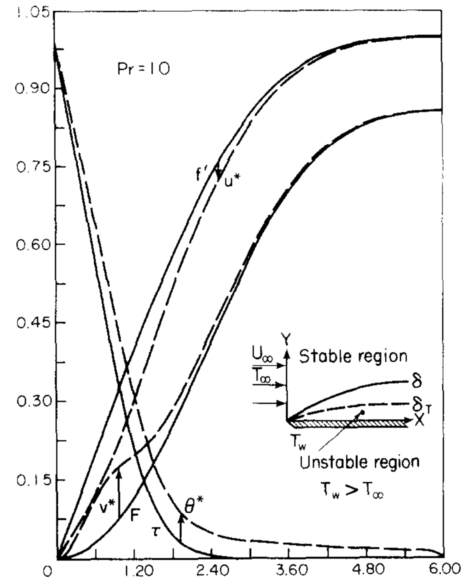

FIG. 2. Distributions of basic quantities f', F, τ and perturbation amplitudes u^*, v^*, θ^* for $Pr = 10$.

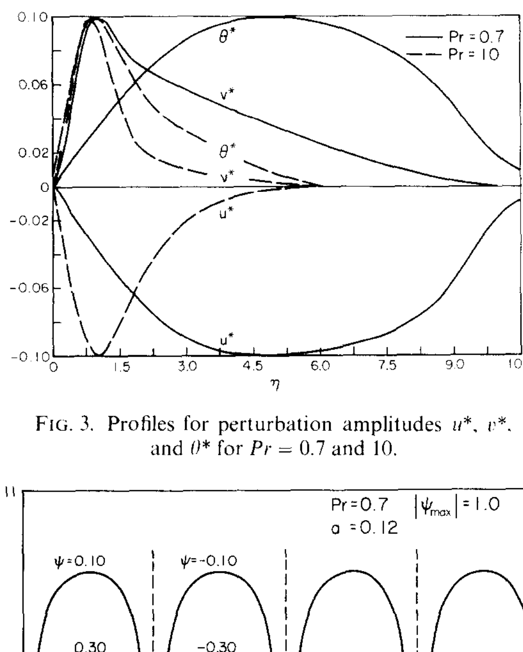

FIG. 3. Profiles for perturbation amplitudes u^*, v^*, and θ^* for $Pr = 0.7$ and 10.

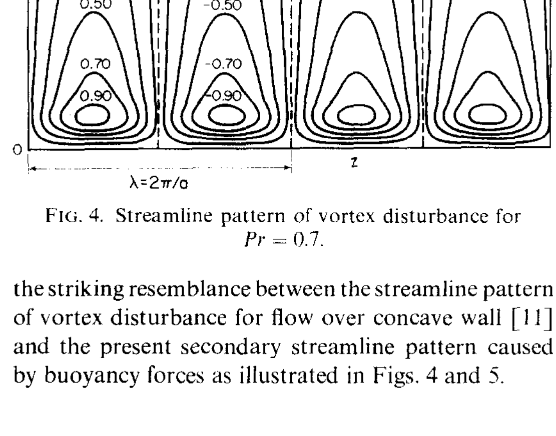

FIG. 4. Streamline pattern of vortex disturbance for $Pr = 0.7$.

Table 2. Numerical result for Gr_L^*

Pr	0.01	0.04	0.06	0.1	0.7	1.0	10	10^2	10^3	10^4
a^*	0.040	0.050	0.050	0.060	0.11	0.14	1.72	2.95	3.90	7.20
Gr_L^*	2472	475.9	360.3	303.9	292.5	270	75.48	13.46	2.406	1.816

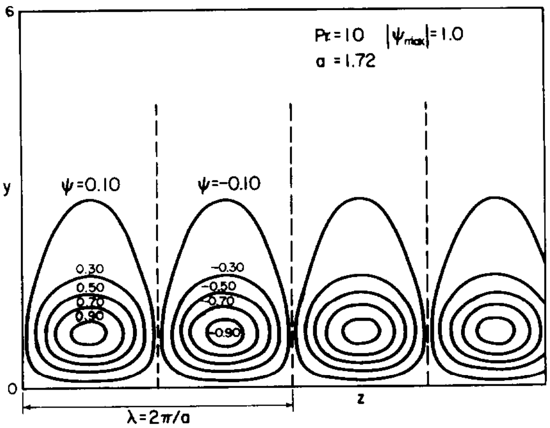

FIG. 5. Streamline pattern of vortex disturbance for $Pr = 10$.

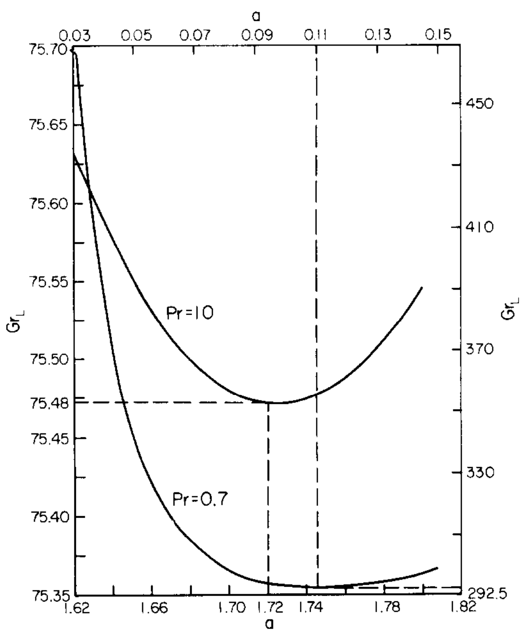

FIG. 6. Neutral stability curves, Gr_L vs a, for $Pr = 0.7$ and 10.

5.2. *The neutral stability results*

The neutral stability curves for Prandtl numbers 0.7 and 10 are presented in Fig. 6 where the eigenvalue Gr_L is plotted against the wavenumber a. The numerical results for the critical (minimum) values of the Grashof number Gr_L^* and the corresponding wavenumber a^* are listed in Table 2 for various Prandtl numbers for future reference and the effect of Prandtl number on the critical Grashof number Gr_L^* is shown in Fig. 7.

Taking cognizance of the relationship $Gr_L = Gr_X/Re_X^{3/2}$, the effect of Reynolds number on the critical Grashof number Gr_X^* can be studied readily and the results are presented in Fig. 8 using logarithmic coordinates. It is of particular interest to compare the present result with Sparrow and Minkowycz's result [2] for 5% increase in local heat-transfer rate due to buoyancy effect based on pure forced convection flow. For this purpose, the curves on Fig. 1 of [2] are also plotted in Fig. 8. To study the implication of the present result, consider the case of Prandtl number 10. The intersection of the two curves for $Pr = 10$ indicates that at $Re_X = 4.8 \times 10^2$, the longitudinal vortices may set in at $Gr_X = 7.9 \times 10^5$. The present results clearly suggest the possible upper limit of the applicability of the published results [1–6]. Figure 8 also shows that for the lower Reynolds number flow, the critical Grashof number Gr_X^* is lower for a given Prandtl number.

CONCLUDING REMARKS

1. The thermal instability of the horizontal Blasius flow heated from below or cooled from above is studied by using linear stability theory based on non-parallel flow model whereby the variations of the basic flow quantities, U_b and T_b, with X as well as the transverse velocity component V_b are retained in the perturbation equations. Some similarity exists between the present problem and the Görtler problem.

2. The result shown in Fig. 7 reveals that the minimum critical value of Gr_L^* is lower for higher Prandtl number. The Prandtl number effect can be explained

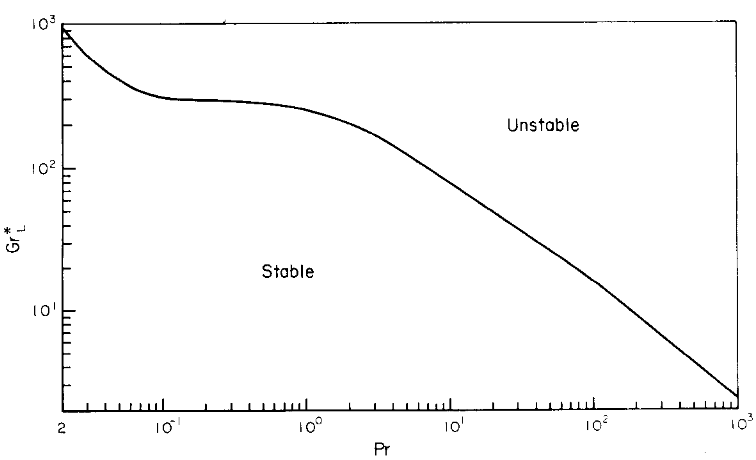

FIG. 7. Relationship between critical Grashof number Gr_L^* and Prandtl number.

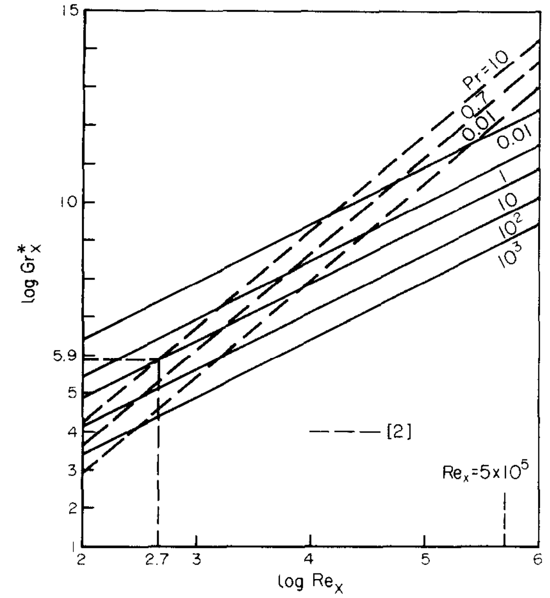

FIG. 8. Critical Grashof number (Gr_X^*)-Reynolds number (Re_X) relation and 5% buoyancy effect on local heat transfer from [2].

from the definition of Gr_L. For the pure laminar forced convection problem the ratio of the thermal boundary-layer thickness over the velocity boundary-layer thickness is known to be $\delta_T/\delta = Pr^{-1/3}$ approximately with $\delta = 5.83(\nu X/U_\infty)^{1/2} = 5.83L$. Noting the above expression and considering the same $\Delta T = T_w - T_\infty$ and Re_X for two different Prandtl numbers, the ratio of the critical Grashof number Gr_L^* can be readily shown to be $(Gr_L^*)_1/(Gr_L^*)_2 = (g\beta/\nu^2)_1/(g\beta/\nu^2)_2$. For example, with $(Pr)_1 = 0.73$ and $(Pr)_2 = 1170$, one finds the ratio $(Gr_L^*)_1/(Gr_L^*)_2 = (4.2\times10^6)/(1.17\times10^4)$ and the order of magnitude checks with the results from the present analysis. It is also noted that the temperature gradient $(\Delta T/\delta_T)$ for large Prandtl number fluid is much larger than that of small Prandtl number fluid. In other words, the unstable region for large Prandtl number fluid is confined to a small region inside the velocity boundary layer. On the other hand, for small Prandtl number fluid, the unstable region extends over a region outside the velocity boundary layer.

3. The basic flow solution for pure forced convection used in this analysis is not valid when Re_X is small (say $< O[10^2]$). For small Re_X, the terms $\partial^2 U_b/\partial X^2$ and $\partial^2 T_b/\partial X^2$ must be included. When $Re_X = 0$, the eigenvalue problem does not exist and a free convection on a heated horizontal semi-infinite flat plate arises. On the other hand, the approximate limit of boundary-layer theory is $Re_X < 5\times10^5$. In interpreting the present results, it must be pointed out that buoyancy effects are considered only in the perturbation equations. An exact analysis would have to consider combined free and forced convection for basic flow. This together with the variable property effect remains to be investigated in future. In this connection, it may be mentioned that the thermal instability analysis considering the buoyancy effects [2] in the basic flow has been completed and the results will be submitted for publication shortly. This remark is added after the completion of the reviewing.

4. The experimental data do not appear to be available for comparison with the present results. It remains for future experiments to obtain the vortex instability data.

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INSTABILITE THERMIQUE DE L'ECOULEMENT DE BLASIUS SUR PLAQUES HORIZONTALES

Résumé—On étudie l'instabilité thermique de l'écoulement de convection forcée laminaire sur une plaque plane horizontale semi-infinie chauffée dans des conditions isothermes par le bas, ou refroidie dans des conditions également isothermes par le haut, pour des perturbations prenant la forme de tourbillons stationnaires longitudinaux et périodiques dans le sens transversal. L'analyse utilise un modèle d'écoulement non parallèle qui tient compte de la variation des champs de vitesse et de température dans la direction longitudinale, ainsi que de la composante de vitesse transversale dans les équations de perturbation. On a obtenu les valeurs critiques du nombre de Grashof $Gr_L = Gr_X/Re_X^{3/2}$ pour des nombres de Prandtl allant de 10^{-2} à 10^{+4}. L'effet des nombres de Reynolds et de Prandtl sur l'instabilité de type tourbillonnaire dans l'écoulement de Blasius sur une plaque horizontale se trouve ainsi éclairci.

THERMISCHE INSTABILITÄT DER BLASIUS-STRÖMUNG ENTLANG WAAGERECHTER PLATTEN

Zusammenfassung—Die thermische Instabilität einer erzwungenen laminaren Konvektionsströmung entlang einer waagerechten, halbunendlichen, ebenen Platte, die isotherm von unten beheizt oder von oben gekühlt ist, wird untersucht. Störungen ergeben sich in Form von stationären Längswirbeln, die in Querrichtung periodisch sind. Die Analysis beruht auf dem Modell nicht-paralleler Strömung unter Berücksichtigung der Veränderung des Grundströmungs- und Temperaturfeldes in Strömungsrichtung sowie einer Komponente der Quergeschwindigkeit in den Störungsgleichungen. Die kritischen Werte der Grashof-Zahl $Gr_L^* = Gr_X^*/Re_X^{3/2}$ wurden erhalten für Prandtl-Zahlen von 10^{-2} bis 10^4. Die Einflüsse der Prandtl- und Reynolds-Zahlen auf die Wirbelinstabilität bei der Blasius-Strömung entlang waagerechter Platten wurden geklärt.

ТЕПЛОВАЯ НЕУСТОЙЧИВОСТЬ ТЕЧЕНИЯ БЛАЗИУСА ВДОЛЬ ГОРИЗОНТАЛЬНЫХ ПЛАСТИН

Аннотация — Исследуется тепловая неустойчивость ламинарного потока в условиях вынужденной конвекции вдоль горизонтальной полубесконечной плоской пластины, изотермически нагреваемой снизу или изотермически охлаждаемой сверху, при возмущении в форме стационарных продольных вихрей, периодически возникающих вдоль потока. Для анализа используется модель непараллельного течения, учитывающая изменение основного потока и температурных полей с изменением продольной координаты, а также поперечной компоненты скорости в уравнениях для возмущений. Получены критические значения числа Грасгофа $Gr_L^* = G_X^*/Re_X^{3/2}$ для числа Прандтля, изменяющегося от 10^{-2} до 10^4. Выявдено влияние чисел Прандтля и Рейнольдса на вихревую неустойчивость для течения Блазиуса вдоль горизонтальной пластины.